

A formula of Ichino on the central value of the L -function of a triple product

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In these notes, K will be a number field and E an étale cubic algebra over K , *i.e.* either $E \simeq K \times K \times K$, or $E \simeq K \times K'$, with K' a quadratic field extension of K , or E is a cubic field extension of K .

1 Definitions and basic properties of triple L -functions

1.1 Continuation and functional equation of triple product L -functions

Given an irreducible automorphic cuspidal representation Π of $\mathrm{GL}_2(\mathbb{A}_E)$, we will consider its L -function $L(\Pi, s)$ as constructed by Piatetski-Shapiro & Rallis in [PR87]. Consider ${}^L G = \hat{G} \rtimes \Gamma$ to be the L -group of $G = R_{E/K} \mathrm{GL}_2$, with $\hat{G} = \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ and $\Gamma = \mathrm{Gal}(\bar{K}/K)$. In the setting of Langlands L -functions, $L(\Pi, s)$ is the L -function associated to the representation r of ${}^L G$ in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ defined so that restriction of r to $\hat{G}$ is the natural eight-dimensional representation and its restriction to Γ acts on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ as the map $\hat{G} \rightarrow \mathfrak{S}_3$ given by E/K . For a careful survey of the normalizing L -factors of the L -function of Π and its adjoint, see the Appendix of [Woo17].

In the setting of three primitive forms f, g, h corresponding to the Galois representations ρ_f, ρ_g, ρ_h , it is classical to consider the L -function associated to $\rho_f \otimes \rho_g \otimes \rho_h$, which is, up to bad local factors a shift of $L(\pi_f \otimes \pi_g \otimes \pi_h, s)$, for π_f, π_g, π_h the automorphic representations associated to f, g, h . The shifting constant depends on the weight of the three forms, and typically for weights k_f, k_g, k_h it will be of the form

$$L(f, g, h, s) \propto L\left(\pi_f \otimes \pi_g \otimes \pi_h, s - \frac{w}{2}\right),$$

with

$$w = \frac{k_f + k_g + k_h - 3}{2}.$$

Generalizing the approach of Garrett [Gar87], Piatetski-Shapiro and Rallis found an integral representation of the L -function of a triple product of irreducible automorphic representations of GL_2 , and used it to study the property of this L -function. From Theorem 5.2 and Theorem 5.3 of [PR87], we can extract the following.

Theorem 1.1 (Garrett, Piatetski-Shapiro & Rallis). *Let Π be an irreducible automorphic cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_E)$. The L -function $L(\Pi, s)$ admits a meromorphic continuation to all $\mathbb{C}$ with a finite number of poles. Moreover, $L(\Pi, s)$ satisfies a functional equation of the form*

$$L(\Pi, s) = \varepsilon(\Pi, s) L(\tilde{\Pi}, 1 - s),$$

where $\varepsilon(\Pi, s)$ is a product of local factors with no zeros or poles.

Remark 1.2. From now on, we will always assume that Π has trivial central character, and under this assumption Π is auto-dual. Then, if $\varepsilon(\Pi, \frac{1}{2}) \neq 1$, then the central value $L(\Pi, \frac{1}{2})$ vanishes trivially. The term $\varepsilon(\Pi, s)$ can be written as the product of local factors $\varepsilon_v(\Pi_v, s) = \pm 1$ (which are uniquely defined in this setting, but not in general), all equal to 1 for all but a finite number of places. Denote by $\Sigma(\Pi)$ the finite set of places at which $\varepsilon_v(\Pi_v, \frac{1}{2}) = -1$ and note that if $\Sigma(\Pi)$ is odd, then the central value vanishes trivially. This discussion will come in handy in Section 3.1.

1.2 First examples of formulae for the central value

Following the work of Garrett and Piatetski-Shapiro & Rallis, Harris & Kudla expressed the central value of the L -function of the triple product of holomorphic newforms over $\mathbb{Q}$.

Theorem 1.3 (Harris & Kudla, [HK04]). *Let $f_i \in \mathcal{S}_{k_i}(\Gamma_0(N_i), \chi_i)$ be holomorphic primitive forms (i.e. cuspidal normalized newforms) for $i = 1, 2, 3$ such that $\chi_1 \chi_2 \chi_3$ is trivial and $k_1 \leq k_2 \leq k_3$. Then $w = k_1 + k_2 + k_3 - 3$ is odd and*

$$I^2 = \frac{L(f_1, f_2, f_3, \frac{w+1}{2})}{p(f_1, f_2, f_3)} \cdot C_\infty \cdot \prod_{p|N_1 N_2 N_3} C_p,$$

with

$$p(f_1, f_2, f_3) = \begin{cases} \langle f_1, f_1 \rangle \langle f_2, f_2 \rangle \langle f_3, f_3 \rangle & \text{if } k_1 + k_2 > k_3 \text{ (balanced)} \\ \langle f_3, f_3 \rangle^3 & \text{if } k_1 + k_2 \leq k_3 \text{ (unbalanced)} \end{cases},$$

where $\langle \cdot, \cdot \rangle$ is the usual Petersson product, with $I \in \mathbb{Q}(f_1, f_2, f_3)$ a global constant, and local constants $C_p \in \mathbb{Q}(f_1, f_2, f_3)^\times$ and

$$C_\infty \in \begin{cases} \pi^{w+2} \mathbb{Q}(f_1, f_2, f_3)^\times & \text{if } k_1 + k_2 > k_3 \\ \pi^{2k_3} \mathbb{Q}(f_1, f_2, f_3)^\times & \text{if } k_1 + k_2 \leq k_3 \end{cases}.$$

Remark 1.4. As it will be explained in Section 3.1, the distinction between balanced and unbalanced triples of modular forms is actually directly related to whether $\varepsilon_\infty(\Pi_\infty, \frac{1}{2})$ is equal to -1 or 1 , where Π denotes the automorphic cuspidal representation associated to the triple modular form. As noted by Harris & Kudla, the balanced case is by far the easiest to handle.

Using the formula from Ichino that will appear below, Collins gave a more explicit formula in the case of almost balanced forms (i.e. when one weight is equal to the sum of the other two) and with minimal hypothesis on the level. He was able to make the constants of the following result completely explicit in the case where both g and h are CM forms.

Theorem 1.5 (Collins, Theorem 2.1.2 of [Col20]). *Let $m > k$ be two fixed positive integers and let $f \in \mathcal{S}_k(\Gamma_1(N_f), \omega_f)$, $g \in \mathcal{S}_{m-k}(\Gamma_1(N_g), \omega_g)$, $h \in \mathcal{S}_m(\Gamma_1(N_h), \omega_h)$ be three holomorphic newforms such that $\omega_f \omega_g = \omega_h$. Let $N = \text{lcm}(N_f, N_g, N_h)$ and fix positive integers M_f, M_g, M_h such that the three integers $M_f N_f$, $M_g N_g$ and $M_h N_h$ divide N and such that none of them is divisible by a larger power of a prime q than the other two. Then*

$$|\langle f_{M_f} g_{M_g}, h_{M_h} \rangle|^2 = \frac{3^2(m-2)!(k-1)!(m-k-1)!}{\pi^{2m+2} 2^{4m-2} M_f^k M_g^{m-k} M_h^m} \cdot L(f, g, h^*, m-1) \cdot \prod_{p|N} C_p,$$

where the C_p 's are local constants and $\varphi_M = \varphi(Mz)$.

Remark 1.6. The convoluted hypothesis on the levels is there to ensure that there is no trivial cancellation of the Petersson product on the left-hand side by avoiding integrating an old form $f_{M_f} g_{M_g}$ against a newform h_{M_h} . This is analogous to choosing the right automorphic forms in the associated automorphic representations.

Example 1.7. *Let N, d be coprime positive integers. In the case of $g \in \mathcal{S}_1(\Gamma_1(d))$, $g^* \in \mathcal{S}_1(\Gamma_1(d))$ (the form whose coefficients of the q -expansion at ∞ are the complex conjugates of the ones of g) and $f \in \mathcal{S}_2(\Gamma_0(N))$ a newform, this gives*

$$|\langle g \cdot g_N^*, f \rangle|^2 = \frac{3^2}{\pi^6 2^6 N} \cdot \prod_{p|Nd} C_p \cdot L(g, g^*, f, 1),$$

2 Quaternion algebras and a conjecture from Jacquet

2.1 Introduction to quaternion algebras

In this section, K will denote a field of characteristic distinct from 2. Quaternion algebras can also be defined in characteristic 2, but the definition is a bit more tedious. See [Voi21, Section 6] or [Vig80] for more details.

Definition 2.1 (Quaternion algebra). A quaternion algebra B of center K is a central algebra of dimension 4 over K such that there exists elements $a, b \in K$ and $i, j \in B$ satisfying the relations

$$i^2 = a, \quad j^2 = b^2, \quad ij = -ji$$

and such that $(1, i, j, ij)$ is a basis of B . The quaternion algebra B is determined by the pair (a, b) up to isomorphism. Additionally, B comes with a K -linear involution $x \mapsto \bar{x}$ defined on its basis by

$$1 \mapsto 1, \quad i \mapsto -i, \quad j \mapsto -j, \quad ij \mapsto -ij.$$

Example 2.2. The algebra $M_2(K)$ of two-by-two matrices with coefficients in K is a quaternion algebra over K for $(a, b) = (1, 1)$, by identifying K to scalar matrices and setting $i = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ and $j = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$.

Example 2.3. Up to isomorphism, the only quaternion algebras over $\mathbb{R}$ are $M_2(\mathbb{R})$ and Hamilton's quaternion algebra $\mathbb{H}$ (corresponding to $(a, b) = (-1, -1)$).

Theorem 2.4 (Classification of quaternion algebras).

- i. A quaternion algebra is either a division algebra or is isomorphic to $M_2(K)$,
- ii. the matrix algebra $M_2(\mathbb{C})$ is the only quaternion algebra over $\mathbb{C}$,
- iii. if K is a local field distinct from $\mathbb{C}$, then there exist precisely two quaternion algebras over K , one of which is $M_2(K)$ and the other is a division algebra,
- iv. if B is a quaternion algebra over a global field K , then for any place v of K , the quaternion algebra B is said to be unramified at v if $B_v := B \otimes_K K_v$ is isomorphic to $M_2(K_v)$ and ramified otherwise (in this case, B_v is a division algebra). The number of places at which B is ramified is finite and even. Moreover, to any set S of places of K of even cardinal corresponds a unique quaternion algebra over K precisely ramified at places contained in S .

A quaternion algebra over $\mathbb{Q}$ is said to be definite if it is ramified at ∞ and indefinite otherwise.

Example 2.5 ([Piz80], Proposition 5.1). Given a prime p , the unique (definite) quaternion algebra ramified at p and ∞ over $\mathbb{Q}$ is given by the pair

- i. $(-1, -1)$ if $p = 2$,
- ii. $(-1, -p)$ if $p \equiv 3 \pmod{4}$,
- iii. $(-2, -p)$ if $p \equiv 5 \pmod{8}$,
- iv. $(-q, -p)$ if $p \equiv 1 \pmod{8}$, where q is a prime number such that $q \equiv 3 \pmod{4}$ and p is not a quadratic residue mod q .

Now, let B be a division quaternion algebra over a number field K . Through the Jacquet-Langlands correspondence, to any automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$ corresponds an automorphic representation of $B^\times(\mathbb{A}_K)$. Precisely, as in [Kna97, Theorem 1.b], let $\mathcal{A}_*(B^\times)$ denote the set of classes of automorphic representations of $B^\times(\mathbb{A}_K)$ with unitary central character that are not one-dimensional and, for any finite set of place S of K , let $\mathcal{A}_0^S(\mathrm{GL}_2)$ denote the set of classes of automorphic representations of $\mathrm{GL}_2(\mathbb{A}_K)$ with unitary central character and whose local components at places in S are of the discrete series.

Theorem 2.6 (Jacquet-Langlands correspondence).

- i. (local correspondence) Let v be a place of K such that $K_v \neq \mathbb{C}$ and let B_v be a division quaternion algebra over K_v (i.e. $B_v \not\cong M_2(K_v)$). To any irreducible admissible representation of $B_v^\times$, we can associate a unique discrete series representation of $GL_2(K_v)$. Moreover, it induces a bijection between classes of irreducible admissible representation of $B_v^\times$ and classes of discrete series representation of $GL_2(K_v)$.
- ii. (global correspondence) Let B be a division quaternion algebra over the number field K and denote by S the set of ramified places of B . To any automorphic representation $\pi^B = \bigotimes_v \pi_v^B$ in $\mathcal{A}_*(B^\times)$, we can associate a unique representation π in $\mathcal{A}_0^S(GL_2)$ such that $\pi_v \cong \pi_v^B$ if $v \notin S$ and for any place $v \in S$, π_v is the local representation associated to π_v^B by the local Jacquet-Langlands correspondence. Furthermore, the map $\pi^B \mapsto \pi$ is a bijection between $\mathcal{A}_*(B^\times)$ and $\mathcal{A}_0^S(GL_2)$.

Remark 2.7. At least two distinct proofs of the correspondence exist. As in [Gel75, Theorem 5.1], one of them uses the Weil representation, while the other uses the trace formula, as exposed in [Kna97, Theorem 1.b].

Remark 2.8. Consider a primitive holomorphic cusp form $f \in \mathcal{S}_k(\Gamma_0(N), \omega)$ and π_f the corresponding irreducible cuspidal automorphic representation. Given a prime p , the form f is said to be p -primitive if its p -part is minimal among all of its twists. If f is p -primitive, then the local non archimedean representation $\pi_{f,p}$ is allowed to be a discrete series only in the case where $p|N$ and $p^2 \nmid N$ (see [LW11]). The local archimedean representation $\pi_{f,\infty}$ is always of the discrete series. Hence, if B is a quaternion algebra over $\mathbb{Q}$ and N is square-free, then there may only exist an automorphic representation π_f^B of $B^\times$ constructed from π_f by the Jacquet-Langlands correspondence if the set of places at which B ramifies is contained in the set of prime factors of N plus the archimedean place. In general, if f is p -primitive then there can exist a local Jacquet-Langlands correspondence with a local ramified quaternion algebra at p only if $p|N$ and $p^2 \nmid N$.

Notation 2.9. For any quaternion algebra B_K over K , let $B_E = B_K \otimes_K E$. Then,

$$B_E^\times = \begin{cases} B_K^\times \times B_K^\times \times B_K^\times & \text{if } E \cong K \times K \times K, \\ GL_2(K') \times B_K^\times & \text{if } E \cong K' \times K \text{ if } K' \text{ is a quadratic field extension of } K, \\ B_{K''}^\times & \text{if } E \cong K'' \text{ is a cubic field extension of } K. \end{cases}$$

If v is a place of K at which B is ramified and Π_v is an admissible representation of $GL_2(K)_E$, then denote by $\Pi_v^{B_v}$ the corresponding representation of $B^\times(\mathbb{A}_E)$ given by the Jacquet-Langlands correspondence if it exists, i.e.

$$\Pi_v^{B_v} \cong \begin{cases} \pi_{1,v}^{B_v} \otimes \pi_{2,v}^{B_v} \otimes \pi_{3,v}^{B_v} & \text{if } E \cong K \times K \times K \text{ and } \Pi_v \cong \pi_{1,v} \otimes \pi_{2,v} \otimes \pi_{3,v}, \\ \pi_{1,v}' \otimes \pi_{2,v}^{B_v} & \text{if } E \cong K' \times K \text{ and } \Pi_v \cong \pi_{1,v}' \otimes \pi_{2,v}, \\ \Pi_v^{B_v} & \text{if } E \cong K'''. \end{cases}$$

2.2 Jacquet's conjecture

Jacquet formulated the following conjecture, relating the vanishing of triple product L -functions to the vanishing of a triple integral over some quaternion algebra.

Conjecture 2.10 (Jacquet). Let π_1, π_2, π_3 be three irreducible cuspidal automorphic representations of $GL_2(\mathbb{A}_K)$ such that the product of their central characters is trivial. Then the central value

$$L(\pi_1 \otimes \pi_2 \otimes \pi_3, \frac{1}{2})$$

does not vanish if and only if there exists a quaternion algebra B over K and automorphic forms $f_i^B \in \pi_i^B$ such that the integral

$$I(f_1^B, f_2^B, f_3^B) = \int_{\mathbb{A}_K^\times B^\times(K) \backslash B^\times(\mathbb{A}_K)} f_1^B(b) f_2^B(b) f_3^B(b) d^\times b$$

does not vanish, where π_i^B is the representation of $B^\times(\mathbb{A}_K)$ associated to π_i by the Jacquet-Langlands correspondence.

Following the work of Garrett and Piatetski-Shapiro & Rallis, Harris and Kudla proved this conjectured in the case where $K = \mathbb{Q}$ and the π_i 's correspond to holomorphic newforms in [HK91], then showed conjecture in full generality in [HK04].

2.3 Gross–Kudla and Böcherer–Schulze-Pillot formulae

Let N be a positive non-square integer, and $f, g, h \in S_2(\Gamma_0(N))$ be three primitive cusp forms. For any prime $p|N$, denote respectively by a_p, b, p, c_p the p -th coefficient of the Fourier expansion at the cusp ∞ of respectively f, g and h . Then, let

$$\varepsilon_p = -a_p b_p c_p$$

and assume that $\#\{p|N \mid \varepsilon_p = -1\}$ is odd. This corresponds to assuming that there is no trivial cancelation of $L(f \otimes g \otimes h, 2)$ due to the functional equation as in this context, $\varepsilon(\Pi, \frac{1}{2})$ is equal to $-\prod_p \varepsilon_p$, cf. [GK92, Proposition 1.1.]. Now, $S = \{p|N \mid \varepsilon_p = -1\} \cup \{\infty\}$ is an even set of places of $\mathbb{Q}$, and it determines a unique definite quaternion algebra ramified at the places in S . Let $R \subset B$ be an order of reduced discriminant N in B , and denote $\hat{R} = R \otimes \hat{\mathbb{Z}}$ and $\hat{B} = B \otimes \hat{\mathbb{Z}}$. Now, let $D/\mathbb{Q}$ be the curve of genus zero associated to B , with a right action of $B^\times(\mathbb{Q})/\mathbb{Q}^\times$, and as in [Gro87, Section 3] and [GK92], let X be the curve defined by

$$X = ((\hat{R}^\times \backslash \hat{B}^\times) \times D)/B^\times,$$

which is the disjoint union of n curves of genus 0 over $\mathbb{Q}$. Thus, the free abelian group $\text{Pic}(X)$ of isomorphism classes of divisors on X has rank n .

As a free abelian group, $\text{Pic}(X)$ is isomorphic to the supersingular module M , which is the free abelian group spanned by the supersingular points of $X_0(\frac{N_0}{p})$ in characteristic p (where $N_0|N$ is the discriminant of B and p is any divisor of N_0). The module $M \otimes \mathbb{C}$ is isomorphic to $S_2(\Gamma_0(N))$ as a Hecke module, so to any newform φ corresponds a line (e_φ) in $\text{Pic}(X) \otimes \mathbb{R}$. This stands as an example of an explicit Jacquet-Langlands correspondence. In [GK92, Corollary 11.3], Gross and Kudla expressed the central value of a triple L -function in terms of a trilinear form in (e_φ) in $\text{Pic}(X) \otimes \mathbb{R}$.

Theorem 2.11 (Gross & Kudla, [GK92]). *Let $f, g, h \in S_2(\Gamma_0(N))$ be three primitive cusp forms, such that $\#\{p|N \mid \varepsilon_p = -1\}$ as defined above is odd. Then*

$$\frac{\langle e_f, e_g, e_h \rangle^2}{\langle e_f, e_f \rangle \langle e_g, e_g \rangle \langle e_h, e_h \rangle} = 2\pi N 2^t \frac{L(f, g, h, 2)}{\langle f, f \rangle \cdot \langle g, g \rangle \cdot \langle h, h \rangle},$$

where t is the number of prime divisors of N , and $\langle \cdot, \cdot, \cdot \rangle$ is a certain trilinear form defined on $\text{Pic}(X) \otimes \mathbb{R}$ and $\langle \cdot, \cdot \rangle$ is a pairing defined on $\text{Pic}(X) \otimes \mathbb{R}$ that is analogue to the Petersson product (and in particular Hecke-equivariant).

In [BS96, Theorem 5.7], Böcherer and Schulze-Pillot generalized the result from Gross and Kudla to a triple primitive cusp forms of possibly distinct level and trivial nebentypes, and possibly distinct balanced weights. This means that if f, g, h have respective weights $k_1 \leq k_2 \leq k_3$, then these weights satisfy $k_1 + k_2 > k_3$. Note that in the work of Gross and Kudla, the weights satisfy $2 + 2 > 2$. This is consistent to the distinction highlighted by Harris and Kudla in Theorem 1.3

While the formula given by Böcherer and Schulze-Pillot is not in the form of the formula given by Gross and Kudla, it can be expressed in this way by replacing the group $\text{Pic}(X)$ by three groups $\text{Pic}(V_f)$, $\text{Pic}(V_g)$, $\text{Pic}(V_h)$, each depending on the level and the weight of one of the three cusp forms, as defined in [Hat90, §3.].

3 Ichino's formula

3.1 Trilinear forms

In [Pra90] and [Pra92], Prasad showed the following result on trilinear forms in most cases, completed by the work of Loke in [Lok01].

Theorem 3.1 (Prasad, Loke). *Let Π be an irreducible admissible representation of $\mathrm{GL}_2(\mathbb{A}_K)$ with trivial central character. Let v be any place of K . Then $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_2(K_v)}(\Pi_v, \mathbb{C}) \leq 1$ and only one of the following two possibilities occurs, namely*

- i. *either $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_2(K_v)}(\Pi_v, \mathbb{C}) = 1$,*
- ii. *or $\Pi_v^{B_v}$ as defined in Notation 2.9 exists and*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{B_v}(\Pi_v^{B_v}, \mathbb{C}) = 1,$$

where B_v is the only division quaternion algebra over K_v .

Moreover, $\varepsilon_v(\Pi_v, \frac{1}{2}) = -1$ if and only if possibility ii. occurs. In other words, $\Sigma(\Pi)$ is exactly the set of places at which $\mathrm{Hom}_{\mathrm{GL}_2(K_v)}(\Pi_v, \mathbb{C})$ is zero. This sums up as a global statement,

- i. *if $\varepsilon(\Pi, \frac{1}{2}) = 1$ (i.e. $\#\Sigma(\Pi)$ is even), then the global quaternion algebra $B = B_{\Sigma(\Pi)}$ over K that ramifies exactly at places in $\Sigma(\Pi)$ is the unique quaternion algebra over K for which $\mathrm{Hom}_{B^\times(\mathbb{A}_K)}(\Pi^B, \mathbb{C})$ is not zero,*
- ii. *if $\varepsilon(\Pi, \frac{1}{2}) = -1$, then $\mathrm{Hom}_{B^\times(\mathbb{A}_K)}(\Pi^B, \mathbb{C}) = 0$ for any quaternion algebra B over K .*

Remark 3.2. Over $\mathbb{Q}$, the theory of definite quaternion algebras (i.e. ramified at the place ∞) is usually easier to conduct than for indefinite ones. It is then natural to expect that explicit formulae will have different nature according to the sign of $\varepsilon_\infty(\Pi_\infty, \frac{1}{2})$. This does not show in Ichino's formula as in some sense it is not explicit, but this distinction is clear in the formula given by Harris and Kudla in Theorem 1.3, as the balance between the weights is reflected in the discrete series representation (or limit of discrete series representation) Π_∞ . Namely, in the setting of Theorem 1.3, the weights are balanced if and only if $\infty \in \Sigma(\Pi)$.

Of course, as a corollary, for any quaternion algebra B over K such that there exist an automorphic representation Π^B of $B^\times(\mathbb{A}_E)$ associated to Π by the Jacquet-Langlands correspondence,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{B^\times(\mathbb{A}_K)}(\Pi^B, \mathbb{C}) \leq 1.$$

Actually, rather than $\mathrm{Hom}_{B^\times(\mathbb{A}_K)}(\Pi^B, \mathbb{C})$, it will be more convenient to use

$$\mathrm{Hom}_{B^\times(\mathbb{A}_K) \times B^\times(\mathbb{A}_K)}(\Pi^B \otimes \tilde{\Pi}^B, \mathbb{C}),$$

which has same dimension over $\mathbb{C}$ that $\mathrm{Hom}_{B^\times(\mathbb{A}_K)}(\Pi^B, \mathbb{C})$.

Since this space has dimension at most one, the most natural bilinear form one could come up with is the element I of $\mathrm{Hom}_{B^\times(\mathbb{A}_K) \times B^\times(\mathbb{A}_K)}(\Pi^B \otimes \tilde{\Pi}^B, \mathbb{C})$ defined for any $\varphi \in \Pi^B$ and $\varphi' \in \tilde{\Pi}^B$ by

$$I(\varphi \otimes \varphi') = \int_{\mathbb{A}_K^\times B^\times(K) \backslash B^\times(\mathbb{A}_K)} \int_{\mathbb{A}_K^\times B^\times(K) \backslash B^\times(\mathbb{A}_K)} \varphi(x) \varphi'(x') d^\times x d^\times x',$$

where $d^\times x$ and $d^\times x'$ both denote the Tamagawa measure on $\mathbb{A}_K^\times \backslash B^\times(\mathbb{A}_K)$.

3.2 Ichino's formula

Let B be any quaternion algebra over K , and assume that there exist an irreducible unitary automorphic representation Π^B of $B^\times(\mathbb{A}_E)$ associated to Π by the Jacquet-Langlands correspondence, e.g. $B = M_2$ and $\Pi^B = \Pi$. In [Ich08], Ichino relates an element of $\mathrm{Hom}_{B^\times(\mathbb{A}_K) \times B^\times(\mathbb{A}_K)}(\Pi^B \otimes \tilde{\Pi}^B, \mathbb{C})$ to the central value of the L -function of Π , where $\tilde{\Pi}^B$ denotes the contragredient representation of $\tilde{\Pi}$. The setting of an automorphic representation over a cubic étale algebra is very specific and rich because of the precise knowledge of trilinear forms, namely by Theorem 3.1.

For each place v of K , denote by I_v the element of $\mathrm{Hom}_{B^\times(K_v) \times B^\times(K_v)}(\Pi^B \otimes \tilde{\Pi}^B, \mathbb{C})$ defined for any $\varphi_v \in \Pi_v^B$ and $\varphi'_v \in \tilde{\Pi}_v^B$ as

$$I_v(\varphi_v \otimes \varphi'_v) = \frac{\zeta_{K_v}(2)}{\zeta_{E_v}(2)} \cdot \frac{L_v(\Pi_v, \mathrm{Ad}, 1)}{L_v(\Pi_v, \frac{1}{2})} \cdot \int_{K_v^\times \backslash B^\times(K_v)} \langle \Pi_v^B(x_v) \varphi_v, \varphi'_v \rangle_v d^\times x_v,$$

where for any v , the $d^\times x_v$ are Haar measures on each $K_v^\times \backslash B^\times(K_v)$ such that $\mathfrak{o}_{K_v}^\times \backslash \mathrm{GL}_2(\mathfrak{o}_{K_v})$ has volume 1 for almost any place v and $\langle \cdot, \cdot \rangle_v$ is any $B^\times(E_v)$ -invariant pairing between Π_v^B and $\tilde{\Pi}_v^B$. Note that there exists a constant C such that

$$d^\times x = C \prod_v d^\times x_v,$$

where $d^\times x$ is the Tamagawa measure on $\mathbb{A}_K^\times \backslash B^\times(\mathbb{A}_K)$.

Thanks to Theorem 3.1 the form I must be a (possibly zero) multiple of the product of the local forms I_v . Ichino's formula makes this relation explicit and states that

Theorem 3.3 (Ichino, [Ich08]). *For any $\varphi = \otimes_v \varphi_v \in \Pi^B$ and $\varphi' = \otimes_v \varphi'_v \in \tilde{\Pi}^B$ so that $\langle \varphi, \varphi' \rangle \neq 0$,*

$$\frac{I(\varphi \otimes \varphi')}{\langle \varphi, \varphi' \rangle} = \frac{C}{2^\alpha} \cdot \frac{\zeta_E(2)}{\zeta_K(2)} \cdot \frac{L(\Pi, \frac{1}{2})}{L(\Pi, \mathrm{Ad}, 1)} \cdot \prod_v \frac{I_v(\varphi_v \otimes \varphi'_v)}{\langle \varphi_v, \varphi'_v \rangle},$$

where $\alpha \in \{1, 2, 3\}$ is the number of components of the étale cubic algebra E and $\langle \cdot, \cdot \rangle$ is the $B^\times(\mathbb{A}_E)$ -invariant pairing between Π^B and $\tilde{\Pi}^B$ defined by

$$\langle \varphi, \varphi' \rangle = \int_{\mathbb{A}_E^\times B^\times(E) \backslash B^\times(\mathbb{A}_E)} \varphi(y) \varphi'(y) d^\times y,$$

with $d^\times y$ the Tamagawa measure on $\mathbb{A}_E^\times \backslash B^\times(\mathbb{A}_E)$. Here, $L(\Pi, \mathrm{Ad}, s)$ denotes the L -function associated to the adjoint representation of ${}^L G$, which is equal to the Rankin-Selberg L -function $L(\Pi \otimes \tilde{\Pi}, s)$, up to bad factors.

Example 3.4 (Collins, [Col20]). *In the case where $K = \mathbb{Q}$, $E = \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ and $B = M_2$, then Ichino's formula can be reformulated as*

$$\frac{\left| \int_{\mathbb{A}_\mathbb{Q}^\times \mathrm{GL}_2(\mathbb{Q}) \backslash B^\times(\mathbb{A}_\mathbb{Q})} \varphi_1(x) \varphi_2(x) \varphi_3(x) d^\times x \right|^2}{\prod_{i=1}^3 \int_{\mathbb{A}_\mathbb{Q}^\times \mathrm{GL}_2(\mathbb{Q}) \backslash B^\times(\mathbb{A}_\mathbb{Q})} |\varphi_i(x)|^2 d^\times x} = \frac{C}{2^3} \cdot \left(\frac{\pi^2}{6} \right)^2 \cdot \frac{L(\pi_1 \otimes \pi_2 \otimes \pi_3, \frac{1}{2})}{\prod_{i=1}^3 L(\pi_i, \mathrm{Ad}, 1)} \prod_v \frac{I_v(\varphi_v \otimes \varphi'_v)}{\langle \varphi_v, \varphi'_v \rangle}.$$

Note that both side could be zero in virtue of Theorem 3.1.

Most of the local trilinear forms I_v are trivial, as done by Ichino in [Ich08, Lemma 2.2.].

Proposition 3.5 (Ichino). *Let v be a non archimedean place of K such that B_v is unramified (i.e. $\Pi_v \cong \Pi_v^B$) and $\mathfrak{o}_{K_v}^\times \backslash \mathrm{GL}_2(\mathfrak{o}_{K_v})$ has volume 1. Then if $\varphi_v \in \Pi_v$ and $\tilde{\varphi}_v \in \tilde{\Pi}_v$ are spherical vectors,*

$$I_v(\varphi_v, \tilde{\varphi}_v) = \langle \varphi_v, \varphi'_v \rangle.$$

Remark 3.6. Note that for any $\varphi \in \tilde{\Pi}$ and $\tilde{\varphi} \in \tilde{\Pi}^B$, the conditions stated in Lemma 3.5 are true for all but a finite number of places of K . The product over all places in Ichino's formula is thus a finite product.

Remark 3.7. In order to find an explicit Ichino formula in a non-automorphic setting, defining the appropriate nontrivial trilinear form and computing the local integrals I_v are the two crucial steps. For example, in Theorem 1.5, the correct trilinear form is the one associated to the product $\langle f_{M_f} \cdot g_{M_h}, h_{M_h} \rangle$. Collins managed to compute the local integrals in the case where f is a newform of trivial nebentype and g, h are CM forms. Many other occurrences of explicit Ichino formulae can be found in the literature, including for Maass forms, as in [Wat08] and [Woo11] in many automorphic cases (including for some Hecke-Maass forms) and as in [Gua24] in the case of the product of a Hecke-Maass form and the adjoint of a CM form.

Remark 3.8. In [MV10, Section 4.4], Michel and Venkatesh showed a formula similar to Ichino's for a triple of standard automorphic representations, when at least one of which is Eisenstein. To ensure convergence of the product of the right-hand side, this implies adding local factors $\zeta_v(1)$. This also implies defining regularized integrals for the left-hand side in the case where all three representations are Eisenstein. In the case where at least one representation is cuspidal, we recover the basic Rankin-Selberg method. Additionally, note that in all cases, since at least representation is Eisenstein, then the relevant quaternion algebra is the nowhere ramified quaternion algebra M_2 . When choosing the right setting, this implies the classical formulae for the central value of double products (Rankin-Selberg, Gross, etc.).

Remark 3.9. In [HY22], Hsieh and Yamana generalized Ichino’s formula in the case where $E = K \times K \times K$ and the character of Π is a quadratic character η . In this case, the ramification of the quaternion algebra must also take into account the sign of $\eta(-1)$.

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